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Evolution of the flow field associated with a streamwise diffusing vortex

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The evolution of a shear flow with an imbedded streamwise vortex is considered. An idealized model for the vortical structure is used; the vortex is assumed to be of infinite extent in the stream direction, and to be a potential vortex (vortex filament) turned on at time zero, and subsequently allowed to diffuse under the action of viscosity. The ambient flow is taken to be, initially, a linear shear profile; the flow then evolves under the joint action of viscosity and convection induced by the vortex. Boundaries are assumed to be infinitely removed from the vortex core. A similarity variable is found which reduces the equation for the induced streamwise velocity perturbation to an ordinary differential equation, which is easily solved numerically. The vortex Reynolds number, circulation/viscosity, is found to be of prime importance. Calculated velocity profiles are presented.

1. Introduction

Many observers have noted the presence of streamwise vortical structures in the buffer region of the turbulent boundary layer; representative papers are Kim, Kline & Reynolds (1971) and Blackwelder & Eckelmann (1979). Associated with these structures are low-speed 'streaks', which have been the subject of intensive experimental scrutiny; a review article by Willmarth (1975) references many of the investigations. Experimental study of boundary-layer transition from laminar to turbulent character (Klebanoff, Tidstrom & Sargent 1962) has given evidence of the association of streamwise vorticity with this process as well; theory (Benney 1961) has shown that growth of mean streamwise vorticity is associated with the nonlinear terms in the form of the Navier–Stokes equations appropriate for a propagating three-dimensional shear wave.

Given the interest in these structures, and their potential relevance to observed phenomena in turbulent boundary layers, it is of some interest to obtain a description of the flow field associated with one of these structures in the absence of any other perturbations to an equilibrium flow field. It turns out that, within the constraints of a few simplifying assumptions, an exact solution of the full Navier–Stokes equations is obtainable in the form of a similarity solution; these assumptions and their implications will form the body of this paper.

2. Assumptions

In any effort directed at the analytical solution of the full Navier–Stokes equations, certain idealizations must be made. For the purposes of this paper, it is assumed that, at time zero, there is no variation of any flow parameter in the x (streamwise) direction; it then follows that there is no x-dependence of any flow parameter at any future time.

Although not rigorously true in any real flow, it is certainly clear from experimental investigations that the streamwise vortical structures are essentially aligned with the flow, and are of very great extent in the streamwise direction; hence it does not seem unreasonable to neglect variations in this direction compared with other terms in the Navier–Stokes equations.

A second assumption is that the undisturbed flow is of the form $U(y) = U_0 + ky$, where U_0 is the velocity at the centre of the vortex, and k is the local shear. Of course, this is only the first-order Taylor-series expansion of the ambient velocity profile about the centreline of the vortex; hence the range of validity for the solution can be no greater than the range of validity of this approximation.

It is further assumed that all solid boundaries are infinitely removed from the centreline of the vortex. This assumption is necessary in order for there to be a similarity solution; on the other hand, if the velocities implied by the model are significant at the position of the (real) wall, then the presence of the wall may alter the flow evolution significantly. Clearly, in any attempt to apply the results presented below to a real flow, the effects of the wall must be somehow estimated in order to establish whether or not this approximation is valid. It is the opinion of the authors that, despite this limitation, the model and its implications are nevertheless of some value.

A final assumption concerns the velocity field at time zero, associated with the vortical structure. For the purposes of this paper, the structure is taken to be a potential vortex filament, inserted into the flow at time zero, which is allowed subsequently to undergo viscous diffusion. The modelling of the vortex structure as a vortex filament (i.e. assuming a $1/r$ tangential velocity dependence) is not uncommon; however, we note in passing that Taylor (1916) proposed a different model for the tangential velocity distribution, which exhibits exponential (rather than $1/r$) decay as $r \rightarrow \infty$. Due to the frequency with which it is appealed to in the literature, we shall be concerned in this paper only with the vortex-filament model; although the Taylor model is also of some interest, it is not amenable to the technique presented below for the filament model, and hence will not be considered further here.

3. Analysis

We are concerned with finding the solution $[u(y,z,t); v(y,z,t); w(y,z,t)]$ which satisfies the incompressible viscous Navier–Stokes equations. In index notation, with $[u = u_1; v = u_2; w = u_3]$ corresponding to the x, y and z velocity components respectively, they are

$$u_{i,t} + u_{i,j}\, u_j = -\frac{1}{\rho} p_{,i} + \nu \Delta u_i, \tag{1}$$

where commas denote partial differentiation, and Δu_i is the Laplacian of u_i in y and z. The flow geometry is shown in figure 1.

Letting primed quantities represent the disturbance to the ambient flow associated with the presence of the vortex, with capitalized quantities representing the ambient (initial) flow field, we have $u_i = u_i' + U_i$. The initial velocity field is taken to be

$$U_1 = U_0 + ky, \quad U_2 = U_3 = 0, \tag{2}$$

where, as stated in §2, U_0 is the velocity at the centreline of the vortex, and k is the local shear.

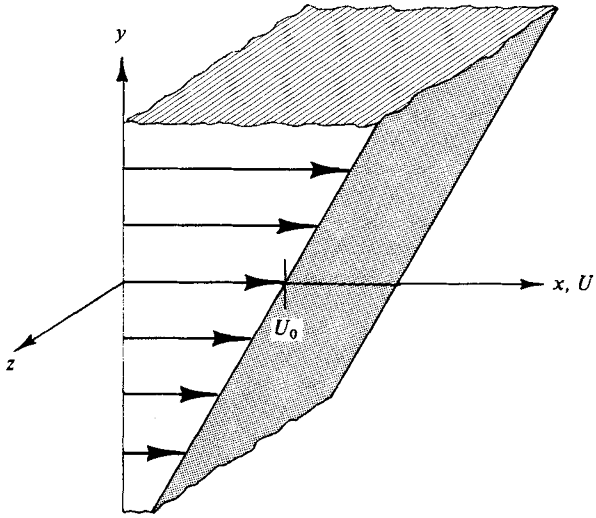

FIGURE 1. Assumed initial flow field $U = U_0 + ky$.

Incompressibility requires

$$u_{i,i} = 0, \tag{3}$$

which, for the assumed form for the U_i, requires

$$u'_{i,i} = 0. \tag{4}$$

By assumption there is no x-dependence in the problem; and as the ambient flow field (2) satisfies (1) and (3) exactly, the equations for the perturbation velocity field become

$$u'_{,t} + (u'_{,y} + k)\, v' + u'_{,z}\, w' = \nu \Delta u', \tag{5}$$

$$v'_{,t} + v'_{,y}\, v' + v'_{,z}\, w' = \nu \Delta v' - \frac{1}{\rho} p'_{,y}, \tag{6a}$$

$$w'_{,t} + w'_{,y}\, v' + w'_{,z}\, w' = \nu \Delta w' - \frac{1}{\rho} p'_{,z}, \tag{6b}$$

$$v'_{,y} + w'_{,z} = 0. \tag{6c}$$

It is evident that (5) and (6) are decoupled, which may at first seem surprising. However, the decoupling is a consequence of the assumed lack of x-dependence in the problem; without x-dependence there can be no net convection of any flow variable in the x-direction. Therefore neither the unperturbed velocity U_1 nor the perturbation streamwise velocity u' appear in the convective terms in (6). It is perhaps useful to note that this decoupling is independent of the assumed form for $U_1(y)$ (or, more generally, $U_1(y, z)$); thus, given a flow field (v', w') that satisfies (6), and boundary conditions, the perturbation streamwise velocity u' may be obtained from

$$u'_{,t} + (u'_{,y} + U_{1,y})\, v' + (u'_{,z} + U_{1,z})\, w' = \nu \Delta u'. \tag{7}$$

This is a linear partial differential equation in the unknown u'; for $U_1 = U_0 + ky$ (7) reduces to (5).

Solution of set (6)

We are, for the purposes of this discussion, assuming that the vortical structure is a potential vortex turned on at time zero, which subsequently undergoes viscous diffusion. We further assume that the boundaries are infinitely removed from the filament. Since the problem evidently exhibits axial symmetry (no angular dependence), it is convenient to solve the set (6) in polar coordinates: we define

$r \equiv$ radial coordinate,
$\theta \equiv$ angular coordinate,
$g' \equiv$ radial velocity,
$h' \equiv$ tangential velocity,
$\mathscr{E}' \equiv$ streamwise vorticity,

Continuity requires

$$\frac{1}{r}\frac{\partial}{\partial r}(rg')+\frac{1}{r}\frac{\partial}{\partial \theta}(h') = 0, \tag{8}$$

which, with axial symmetry, implies

$$rg' = C;$$

boundedness as $r \to 0$ then requires $C = 0$, which gives

$$g' = 0.$$

The streamwise vorticity, in terms of g' and h', is

$$\mathscr{E}' = \frac{1}{r}\frac{\partial}{\partial r}(rh')-\frac{1}{r}\frac{\partial g'}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}(rh'), \tag{9}$$

and satisfies the equation†

$$\mathscr{E}'_{,t}+g'\mathscr{E}'_{,r}+\frac{h'}{r}\mathscr{E}'_{,\theta} = \nu\left(\mathscr{E}'_{,rr}+\frac{1}{r}\mathscr{E}'_{,r}+\frac{1}{r^2}\mathscr{E}'_{,\theta\theta}\right), \tag{10}$$

which in this case reduces to

$$\mathscr{E}'_{,t} = \nu\left(\mathscr{E}'_{,rr}+\frac{1}{r}\mathscr{E}'_{,r}\right), \tag{11a}$$

which is the radial diffusion equation. For a filament turned on at time zero the appropriate initial condition is

$$\mathscr{E}'(r,t=0) = \frac{\Gamma}{2\pi r}\delta(r), \tag{11b}$$

where Γ is the circulation associated with the vortex, and $\delta(r)$ is the Dirac delta function.

† The vortex stretching and tilting terms which would normally appear on the right-hand side of (10) can be seen to have zero net contribution as follows. If $\psi' \equiv \partial g'/\partial x-\partial u'/\partial r \equiv$ tangential vorticity, and $\phi' \equiv (1/r)\,\partial u'/\partial\theta-\partial h'/\partial x \equiv$ radial vorticity, then the right-hand side of the streamwise vorticity equation should have the additional terms $\mathscr{E}'\,\partial u'/\partial x+\phi'\,\partial u'/\partial r+(\psi'/r)\,\partial u'/\partial\theta$. For no x-variation we obtain $\phi' = (1/r)\,\partial u'/\partial\theta$, $\psi' = -\partial u'/\partial r$ and $\partial\mathscr{E}'/\partial x = 0$; hence the surviving terms $\phi'\,\partial u'/\partial r+(1/r)\,\psi'\,\partial u'/\partial\theta \equiv 0$ identically. Note that axial symmetry is not required for this result.

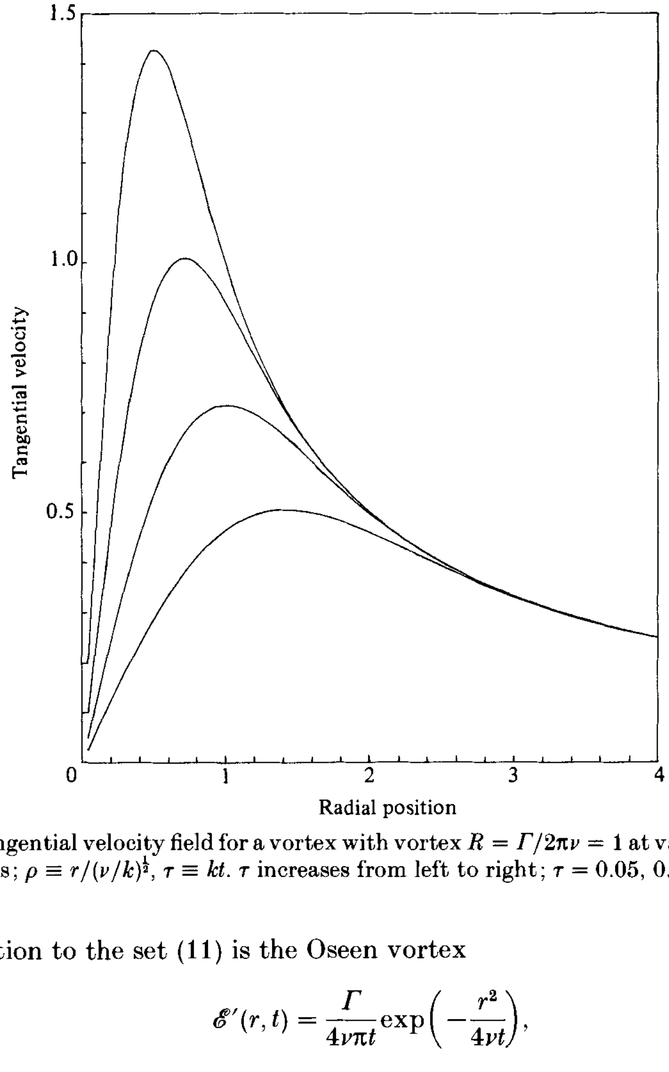

FIGURE 2. Tangential velocity field for a vortex with vortex $R = \Gamma/2\pi\nu = 1$ at various dimensionless times; $\rho \equiv r/(\nu/k)^{\frac{1}{2}}$, $\tau \equiv kt$. τ increases from left to right; $\tau = 0.05, 0.1, 0.2, 0.4$.

The solution to the set (11) is the Oseen vortex

$$\mathscr{E}'(r,t) = \frac{\Gamma}{4\nu\pi t}\exp\left(-\frac{r^2}{4\nu t}\right), \tag{12}$$

with associated tangential velocity field

$$h'(r,t) = \frac{\Gamma}{2\pi r}\left(1-\exp\left(-\frac{r^2}{4\nu t}\right)\right), \tag{13}$$

which is obtained by substituting (12) into (9) and integrating; a plot of $h'(r,t)$ for various values of t and a particular Γ, suitably non-dimensionalized as described below, is presented in figure 2.

We now have in hand a complete solution to the set (6); it remains only to utilize this solution to generate the solution $u'(y,z,t)$ to (5).

Equation (5) is also best considered in polar coordinates; in these coordinates the equation is

$$u'_{,t}+\frac{h'}{r}(U+u')_{,\theta} = \nu\left(u'_{,rr}+\frac{1}{r}u'_{,r}+\frac{1}{r^2}u'_{,\theta\theta}\right), \tag{14}$$

where h' is given by (13), and use has been made of the fact that there is no radial component of velocity.

It is convenient to non-dimensionalize (14); since the boundaries are infinitely removed from the vortex centreline there is no imposed physical lengthscale. There is, however, a characteristic time $T = 1/k$ associated with the local shear $\mathrm{d}U_1/\mathrm{d}y = k$; a viscous lengthscale is then $L = (\nu/k)^{\frac{1}{2}}$. The characteristic velocity is then $V = (\nu k)^{\frac{1}{2}}$.

Setting $\bar{u} \equiv$ non-dimensional streamwise perturbation velocity; $\bar{U} \equiv$ dimensionless ambient velocity; $\rho \equiv$ dimensionless radial displacement and $\tau \equiv$ dimensionless time, we obtain

$$\bar{u}_{,\tau}+\frac{\Gamma}{2\pi\nu}\frac{1}{\rho^2}\left(1-\exp\left(-\frac{\rho^2}{4\tau}\right)\right)(\bar{u}+\bar{U})_{,\theta} = \bar{u}_{,\rho\rho}+\frac{1}{\rho}\bar{u}_{,\rho}+\frac{1}{\rho^2}\bar{u}_{,\theta\theta}. \tag{15}$$

In the non-dimensionalization the ambient velocity becomes

$$\bar{U} = \bar{U}_0+\rho\sin\theta.$$

Clearly the only parameter appearing in (15) is the ratio $\Gamma/2\pi\nu$, a measure of the relative importance of circulation (convection) and viscous effects in the flow evolution. This can be thought of as a Reynolds number for the problem; in the subsequent discussion it is referred to as the vortex Reynolds number R.

Using the assumed form for $\bar{U}$, we obtain

$$\bar{u}_{,\tau}+\frac{R}{\rho^2}\left(1-\exp\left(-\frac{\rho^2}{4\tau}\right)\right)\bar{u}_{,\theta}-\bar{u}_{,\rho\rho}-\frac{1}{\rho}\bar{u}_{,\rho}-\frac{1}{\rho^2}\bar{u}_{,\theta\theta} = -\frac{R}{\rho}\left(1-\exp\left(-\frac{\rho^2}{4\tau}\right)\right)\cos\theta. \tag{16a}$$

The perturbation velocity must be initially zero everywhere, since the vortex is 'turned on' at time zero; furthermore, symmetry considerations require that there be no streamwise velocity perturbation at the vortex core ($r = 0$). Finally, there can be no perturbation streamwise velocity at infinity; these considerations lead to the boundary conditions

$$\bar{u}(0,\theta,\tau) = 0, \quad \bar{u}(\infty,\theta,\tau) = 0, \quad \bar{u}(\rho,\theta,0) = 0. \tag{16b}$$

This is a well-posed problem of the diffusion type; it is soluble in a rather straightforward way numerically, marching forward in time, and in fact this was the approach we adopted initially.

Fortunately a more illuminating approach is available; this consists of defining

$$\rho f(\rho,\theta,\tau) = \bar{u}(\rho,\theta,\tau). \tag{17a}$$

It turns out that $f(\rho,\theta,\tau)$ is expressible as $f(\eta,\theta)$ only, where the similarity variable $\eta \equiv \rho^2/4\tau$. Furthermore, it is possible to write

$$f(\eta,\theta) = \mathrm{Re}\,(F(\eta)\,\mathrm{e}^{\mathrm{i}\theta}); \tag{17b}$$

these transformations then reduce the linear partial differential equation (16a) to the solution of a linear ordinary differential equation, with complex coefficients, in the similarity variable η.

After substitution and rearrangement, (16) becomes

$$4\eta^2F''+(4\eta^2+8\eta)\,F'-\mathrm{i}R(1-\exp(-\eta))\,F = R(1-\exp(-\eta)). \tag{18}$$

Although not evidently representable in terms of known functions, the (complex) solution $F(\eta)$ can be obtained numerically with relative ease (e.g. by finite differences) once the boundary conditions at $\eta = 0$ and $\eta = \infty$, corresponding to (16b), can be found.

The boundary conditions on $F(\eta)$ require a bit of care, since they are to be imposed at $\eta = 0$ (a regular singular point of (18)) and at $\eta = \infty$ (an irregular singular point of (18)). Near $\eta = 0$, there is guaranteed to be at least one homogeneous solution of the form

$$F_1^{(h)} = \eta^{\alpha}\left(\sum_{j=0}^{\infty} a_j \eta^j\right); \tag{19}$$

substituting in the assumed form into the homogeneous counterpart to (18) gives

$$\alpha_1 = 0, \quad \alpha_2 = -1$$

as the potential solutions. Since these roots differ by an integer, we are only assured of a solution of the form (19) for the larger value α_1; a second independent homogeneous solution has the form

$$F_2^{(h)} = F_1^{(h)} \ln \eta + \eta^{-1} \sum_{j=0}^{\infty} b_j \eta^j. \tag{20}$$

A particular solution to (18) near $\eta = 0$ is obtained by expanding the exponential terms which appear in powers of η; we find that there also exists a particular solution of Taylor-series form. Requiring that the solution $F(\eta)$ to (18) be bounded at the origin (note that if $F(\eta)$ is bounded, then $\bar{u}|_{\rho=0} = \rho[\mathrm{Re}\,(F\mathrm{e}^{\mathrm{i}\theta})|_{\eta=0}] = 0$) we are justified in seeking a full solution of the form

$$F(\eta) = \sum_{j=0}^{\infty} c_j \eta^j.$$

Substituting this form into (18) directly, the resulting recursion relationship for the c_k is

$$4k(k+1)\,c_k + 4(k-1)\,c_{k-1} + \mathrm{i}R \sum_{n=1}^{k-1} c_n(-1)^{k-n} \frac{1}{(k-n)!} = R(1+\mathrm{i}c_0)\frac{(-1)^{k+1}}{k!} \quad (k \geqslant 1). \tag{21}$$

Note that, while c_0 is arbitrary, for any given c_0 the remaining coefficients are completely determined; in particular

$$c_1 = \tfrac{1}{8}R(1+\mathrm{i}c_0). \tag{22a}$$

Thus the requirement that $F(\eta = 0)$ be bounded yields a unique relationship between $F|_{\eta=0}$ and $\mathrm{d}F/\mathrm{d}\eta|_{\eta=0}$.

For the point at infinity, in order to have $\bar{u} = \rho\,\mathrm{Re}\,(F\mathrm{e}^{\mathrm{i}\theta})$ go to zero, $F(\eta)$ must decay faster than $\eta^{-\frac{1}{2}}(= 2\tau^{\frac{1}{2}}/\rho)$. To show that this in fact imposes a constraint on the system, it is necessary to consider the asymptotic behaviour of F as $\eta \to \infty$. Representing F as a sum of particular and homogeneous solutions asymptotically valid as $\eta \to \infty$,

$$F \sim F^{\mathrm{p}}(\eta) + d_1 F_3^{(\mathrm{h})}(\eta) + d_2 F_4^{(\mathrm{h})}(\eta) \quad \text{as } \eta \to \infty,$$

where $F_3^{(\mathrm{h})}$ and $F_4^{(\mathrm{h})}$ are linearly independent asymptotic homogeneous solutions to (18). It is not difficult to show that

$$F^{(\mathrm{p})} \sim -\frac{R}{4\eta}\left(1 - \frac{\mathrm{i}R}{8\eta} + O(\eta^{-2})\right), \quad F_3^{(\mathrm{h})} \sim \frac{1}{\eta^2}\mathrm{e}^{-\eta}\left(1 + \frac{\mathrm{i}R-8}{4\eta} + O(\eta^{-2})\right),$$

$$F_4^{(\mathrm{h})} \sim 1 - \frac{\mathrm{i}R}{4\eta} + O(\eta^{-2})$$

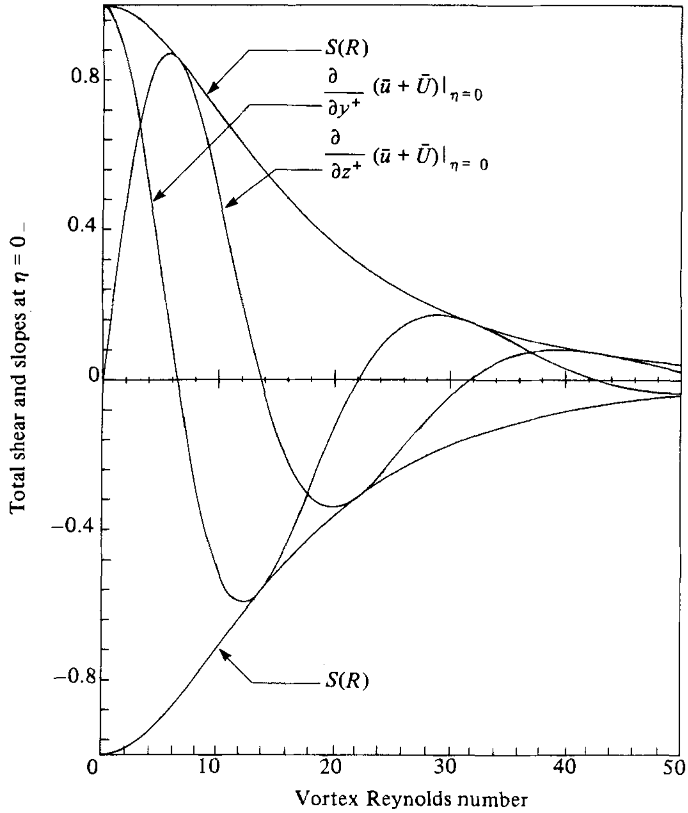

FIGURE 3. Plot of (dimensionless) $|\nabla(\bar{u}+\bar{U})| \equiv S(R, \eta = 0)$ in 'core' region of vortex ($\eta \rightarrow 0$) against vortex Reynolds number; $\partial(\bar{u}+\bar{U})/\partial y|_{\eta=0}$ and $\partial(\bar{u}+\bar{U})/\partial z|_{\eta=0}$ are also plotted.

are allowable choices for the particular solution and the two independent homogeneous solutions respectively. The requirement that F decay sufficiently rapidly as $\eta \rightarrow \infty$ is thus seen to be equivalent to the requirement that there be no contribution from the $F_4^{(h)}$ function to the complete solution F; hence, requiring $F \rightarrow 0$ as $\eta \rightarrow \infty$ does indeed provide the second constraint (in addition to (22a) needed to uniquely determine a solution to (18). To recapitulate: the non-dimensional perturbation velocity $\bar{u}$ that exactly solves (16) – and hence the full Navier–Stokes equations – is

$$\bar{u} = \mathrm{Re}\,(\rho F(\eta)\, \mathrm{e}^{\mathrm{i}\theta}),$$

where $F(\eta)$ is the unique solution to

$$4\eta^2 F'' + (4\eta^2 + 8\eta)\, F' - \mathrm{i}R(1-\exp(-\eta))\, F = R(1-\exp(-\eta)), \tag{18}$$

with boundary conditions

$$F'(0) - \tfrac{1}{8}\mathrm{i}R\, F(0) = \tfrac{1}{8}R,$$
$$F(\infty) = 0. \qquad (22a, b)$$

In the numerical solution, the semi-infinite domain $0 \leqslant \eta \leqslant \infty$ is mapped into $0 \leqslant \chi \leqslant 1$ via the transformation $\chi = \eta/(\eta+1)$; this makes the condition at infinity somewhat easier to impose.

A few comments on the solutions associated with these equations are in order.

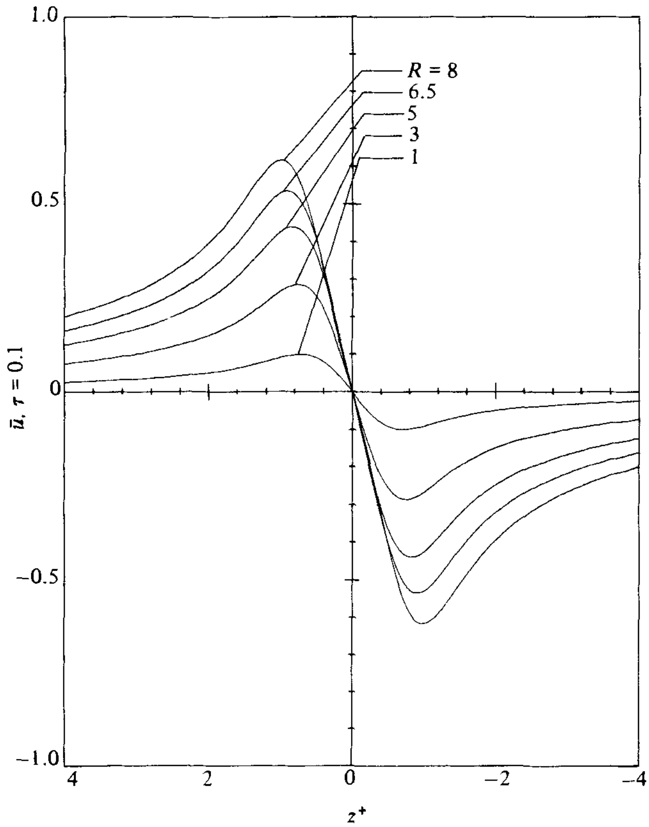

FIGURE 4. Plot of dimensionless perturbation streamwise velocity $\bar{u} = u'/(\nu k)^{\frac{1}{2}}$ against spanwise position $z^+ \equiv z/(\nu/k)^{\frac{1}{2}}$, for vortex $R = 1, 3, 5, 6.5, 8$, at dimensionless time $\tau \equiv kt = 0.1$, and $y^+ = 0$.

Since as $\tau \to \infty$, $\eta \to 0$ for any fixed radial position ρ, we have

$$\begin{aligned} \lim_{\tau\to\infty} \bar{u}(\rho,\theta,\tau) &= \mathrm{Re}\,[\rho F(0)\,\mathrm{e}^{\mathrm{i}\theta}] \\ &= \mathrm{Re}\,(F(0))\,\rho\,\cos\theta - \mathrm{Im}\,F(0))\,\rho\,\sin\theta \\ &= \mathrm{Re}\,(F(0))\,(-z^+) - \mathrm{Im}\,(F(0))\,y^+, \end{aligned}$$

where $z^+ \equiv$ dimensionless spanwise position $\equiv z(k/\nu)^{\frac{1}{2}}$ and $y^+ \equiv y(k/\nu)^{\frac{1}{2}}$.

In other words, the effect of the vortex is to establish a new equilibrium shear profile within a 'core' region, the radial extent of which scales with $\tau^{\frac{1}{2}}$. In fact, the total (dimensionless) velocity field inside the core region is, with

$$C_0^{(\mathrm{i})} \equiv \mathrm{Im}\,(F(0)) \quad \text{and} \quad C_0^{(\mathrm{r})} \equiv \mathrm{Re}\,(F(0)), \quad \lim_{\eta\to 0} \bar{U} + \bar{u} = (1 - C_0^{(\mathrm{i})})\,y^+ - C_0^{(\mathrm{r})}\,z^+. \tag{23}$$

A measure of the mixing associated with the presence of the vortex is the quantity

$$S(R) \equiv \left[\left(\frac{\partial(\bar{u}+\bar{U})}{\partial y^+}\right)^2 + \left(\frac{\partial(\bar{u}+\bar{U})}{\partial z^+}\right)^2\right]^{\frac{1}{2}},$$

where R is the vortex Reynolds number, and hence

$$S(R) = [(1 - C_0^{(\mathrm{i})})^2 + (C_0^{(\mathrm{r})})^2]^{\frac{1}{2}} \tag{23a}$$

inside the core region.

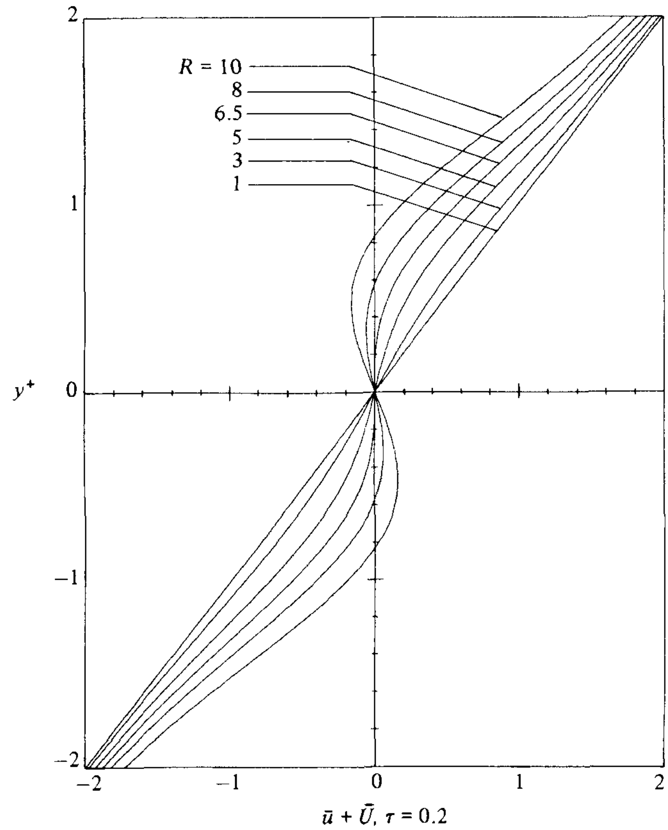

FIGURE 5. Plot of dimensionless streamwise velocity (ambient+perturbation) $\bar{u}+\bar{U}$ against cross-stream position $y^+ \equiv y/(\nu/k)^{\frac{1}{2}}$, for vortex $R = 1$, 3, 5, 6.5, 8 and 10, at $\tau = 0.2$, and $z^+ = 0$. $R = 1$ is essentially indistinguishable from the unperturbed ($R = 0$) profile.

Here, of course, $S(R)$ is the magnitude of the (dimensionless) gradient in the core ($\eta \ll 1$) region; the unperturbed flow ($\tau = 0$) has a dimensionless gradient of magnitude 1. A plot of $S(R)$ is given in figure 3; note that as $R \to \infty$, $S \to 0$, indicating that, for large R, there is essentially complete mixing.

Another interesting feature of these equations is the catastrophic effect of the vortex on the ambient profile, even at modest vortex Reynolds numbers. Instantaneous profiles for the total streamwise velocity $\bar{u}+\bar{U}$ along the y^+ coordinate axis, and for the perturbation velocity $\bar{u}$ along the z^+ coordinate axis, are presented for a range of vortex Reynolds numbers in figures 4 and 5; profiles along the y^+ axis for a specific vortex Reynolds number ($R = 5$) at various values of dimensionless time are displayed in figure 6. The severity of the inflection in the streamwise velocity profile even at a vortex $R = 10$ (which implies a dimensional circulation of 0.6 cm^2/s for water) is truly remarkable.

4. Discussion

The simplicity of the results presented in § 3, and the obvious physical interpretations of the results, is clearly an attractive feature. The model yields a revealing picture of the physical balances inside a streamwise diffusing vortex; the primary effect of

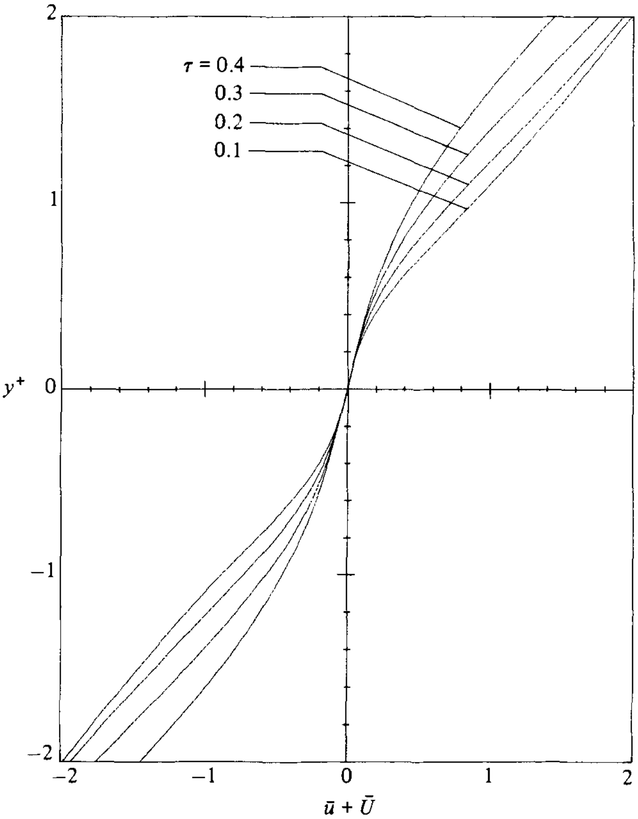

FIGURE 6. Plot of dimensionless stream velocity $\bar{u}+\bar{U}$ against cross-stream position y^+, for vortex $R = 5$, at $\tau = 0.1, 0.2, 0.4$ and 0.8, and $z^+ = 0$. Note temporal development of (locally) linear 'core' region.

the vortex is to redistribute momentum between regions of high and low momentum (respectively above and below the centreline). Mitigating this effect are the viscous forces; these lead to the re-establishment of a (locally) linear shear flow, but with reduced total shear $S = |\nabla u|$. It is also interesting to note that the predictions of streamwise velocity perturbations associated with vortices of known circulation gives one a potentially useful inverse method for finding a characteristic vortex strength associated with measured perturbations to the streamwise velocity; for slowly spinning structures this could well prove to be a more sensitive technique than direct measurement, where noise can become a significant problem.

Of course, this new shear profile is steady state only in the absence of external forces, such as a pressure gradient in the stream direction; the presence of the pressure gradient would work to re-establish the original equilibrium profile. It is important to emphasize that in the model presented here the flow will not return to its original equilibrium state; there is no restoring force in this model. However, in the presence of a body force or pressure force, the above results would still be valid for sufficiently small values of time and radial displacement; this is because the vortex must 'disequilibrate' the flow before restoring forces can begin to push the flow back towards equilibrium.

Since the solution is a similarity solution, restricting the validity of the solution in this way in no way changes the basic result, which is the development of kinked velocity profiles in response to the presence of the vortex; one would expect, however, that the similarity solution would become a poor approximation as the integrated acceleration due to external force imbalances becomes appreciable, i.e. as time becomes large.

The restriction to small radial displacements arises similarly from deviations in a real flow field from the conditions assumed in the model, e.g. the presence of walls and nonlinear mean-velocity profiles. We argue that these effects will also be negligible for small r and t, as the velocity profile will be approximately linear in r about the vortex core, and the walls will only communicate their presence on some diffusive timescale.

It therefore seems eminently plausible that the model yields correct predictions for the small-r small-t regime; it remains to consider the dynamical instabilities in the velocity profile induced by the vortex. Although the calculations presented in §3 yield an exact time-dependent solution to the Navier–Stokes equations, it may well be that slight imperfections in the ambient flow, or deviations from the assumptions in the model, might induce exponential divergence from the calculated solution. To answer this question, one must perform the full three-dimensional stability of the two-dimensional time-dependent flow field; efforts towards calculating the linear stability, both theoretically and experimentally, using a vibrating-ribbon technique, are currently in progress. One might note in passing that, even though the profiles become increasingly inflectional with increasing vortex Reynolds number, it is not possible to make any statements concerning the stability based on Rayleigh's theorem, since that result is only valid for a bounded two-dimensional flow; however, Rayleigh's theorem does lead one to suspect that, at least for sufficiently high vortex Reynolds numbers, one will find manifestation of dynamical instability.

5. Conclusion

An extraordinarily simple solution to the full Navier–Stokes equations has been found for the case of a diffusing, initially potential vortex (Oseen vortex), aligned with the stream, and imbedded in an infinite linear shear flow. The solution potentially can be used as a tool for experimental work on vortex structures, although the solution should not be considered valid far away from the vortex centreline. Strongly inflectional streamwise velocity profiles for even relatively modest circulations raise the question of stability; as vortical structures are observed in turbulent boundary layers, this question might prove to be of some importance. Efforts directed towards its solution are currently underway.

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